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ON SOME GEOMETRIC PROPERTIES OF CIRCULAR TRANSFORMATIONS.

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§ 1. **Introduction.** In what follows I shall give simple demonstrations of a few geometric theorems that are of value in the study of ring-shaped domains as used in the definition of certain special automorphic functions.[1] As will be seen later on, the theory of such domains is closely connected with linear substitutions or circular transformations in a complex plane, as defined in § 3. Although there is nothing essentially new as to content, it will be at once apparent, how simply, in comparison with ordinary analytic methods, some of the theorems may be proved by making use of a fundamental proposition in group-theory. With this fact I want to emphasize the importance of an early introduction of linear substitutions and their principal group-properties in certain mathematical courses, as for example in function-theory.

A *ring-shaped domain* G may be defined as a connected portion of a complex plane bounded by two non-intersecting circles (see Fig. p. 140). Among such domains are included those into which G passes when the two circles become tangent.

By a *group* we understand a class of operations, such that the product of any two operations of the class is again an operation of the same class. The interpretation of this definition for linear substitutions will be found in § 3.

§ 2. **Reflexion on a Circle.** By *reflexion* of a point P on *a straight line* l we define a point P' on a line through P perpendicular to l, such that the distances of the distinct points P and P' from l are the same. The transformation by reciprocal radii with respect to a fixed circle is called *reflexion on the circle.*

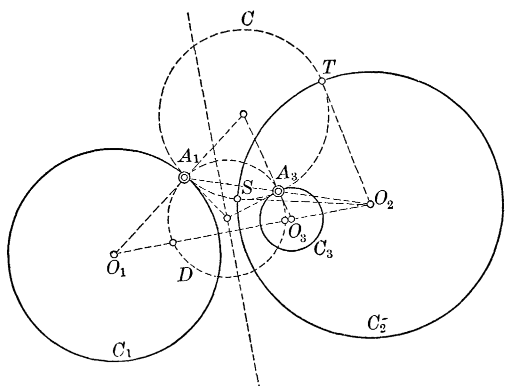
An *inversion* with respect to a fixed circle may be defined as a reflexion on that circle followed by a reflexion on a fixed axis through the center of the same circle. This circle may, of course, be located anywhere in the plane.

[1] SCHOTTKY: "Ueber die Funktionenklasse die der Gleichung $F(\alpha x + \beta/\gamma x + \delta) = F(x)$ genügt," *Crelle's Journal*, Vol. 143, pp. 1–24 (June, 1913). See also the same *Journal*, Vol. 101, pp. 231–236 (1887).

Two circles C_1 and C_3 determine a *pencil of circles*, by which we understand the totality of circles passing through the points of intersection of C_1 and C_3. The pencil is called *hyperbolic* when the points of intersection are real and distinct; *parabolic*, when they are coincident; *elliptic*, when they are imaginary. Denoting the radii of C_1 and C_3 by r_1 and r_3, and the distance between their centers by e, the distance ϵ between the null-circles of the pencil is easily found to be

$$\epsilon = \frac{2r_1r_3}{e}\sqrt{\frac{(r_1^2 + r_3^2 - e^2)^2}{4r_1^2r_3^2} - 1}. \tag{1}$$

This expression is real, vanishes, or is imaginary, according as C_1 and C_3 do not



intersect (in real points), are tangent, or do intersect. The angle θ at which the two circles intersect is defined by

$$\sin^2\theta = \frac{(r_1^2 + r_3^2 - e^2)^2}{4r_1^2r_3^2} - 1 \tag{2}$$

and is imaginary in the elliptic, vanishes in the parabolic, and is real in the hyperbolic case. This expression for $\sin^2\theta$ is important in a certain proof of convergence given by Schottky, *loc. cit.*

A circle C_2 may now be constructed so that C_1 and C_3 are reflexions of each other on C_2. Every circle C tangent to both C_1 and C_3 at points A_1 and A_3, which are reflexions on C_2, cuts this circle orthogonally. See Fig. By any other reflexion on a circle L, the circles C_1, C_2, C_3, C are transformed into circles C_1', C_2', C_3', C', so that all circles C' are orthogonal to C_2'. Consequently C_1' and C_3' are reflexions on C_2'. We may state this as

THEOREM 1. *By a reflexion on a circle L three circles C_1, C_2, C_3 of which C_1 and C_3 are reflexions on C_2 are transformed in the same order into three circles C_1', C_2', C_3' of which C_1' and C_3' are reflexions on C_2'.*[1]

When the center of L is on C_2, then C_2 is transformed into a straight line C_2', and C_1' and C_3' become equal circles and are ordinary reflexions on C_2'.

[1] This, in another form, corresponds to theorem 3, p. 241, of *Osgood's Lehrbuch der Funktionentheorie*, Vol. 1, 2d ed. (1912).

An elliptic pencil with the origin and a point on the real axis at a distance e as limiting points may be represented by

$$(x - e)^2 + y^2 - \lambda(x^2 + y^2) = 0$$

or

$$\left(x - \frac{e}{1 - \lambda}\right)^2 + y^2 = \frac{\lambda e^2}{(1 - \lambda)^2}, \tag{3}$$

for all real positive values of λ. For a definite λ, the distances of the limiting points from the center of the circle (3) are

$$\frac{e}{1 - \lambda} \quad \text{and} \quad \frac{e}{1 - \lambda} - e = \frac{\lambda e}{1 - \lambda}.$$

The product of the two distances, $\lambda e^2/(1 - \lambda)^2$, equals the square of the radius of the circle (3). From this it follows immediately that *the limiting points of an elliptic pencil are reflexions on every circle of the pencil* and we have

THEOREM 2. *A reflexion on any circle of an elliptic pencil transforms the pencil into itself. The same is, of course, also true of an hyperbolic and a parabolic pencil, since all circles of a pencil pass through two fixed points, which are imaginary in an elliptic, real and distinct in an hyperbolic and coincident in a parabolic pencil.*

In the circle (3) the distances of the extremities of the diameter on the axis from the origin, one of the limiting points, are

$$\frac{e}{1 - \lambda} + \frac{e\sqrt{\lambda}}{1 - \lambda} = \frac{e}{1 - \lambda}(1 + \sqrt{\lambda}),$$

and

$$\frac{e}{1 - \lambda} - \frac{e\sqrt{\lambda}}{1 - \lambda} = \frac{e}{1 - \lambda}(1 - \sqrt{\lambda}).$$

Inverting this we get for the center of the reflected circle (3) on the unit-circle around the origin the distance

$$\frac{1}{2}\left\{\frac{1 - \lambda}{e(1 + \sqrt{\lambda})} + \frac{1 - \lambda}{e(1 - \sqrt{\lambda})}\right\} = \frac{1}{e},$$

an expression which is independent of λ. Hence

THEOREM 3. *The reflexion of an elliptic pencil on a circle having one of the limiting points as a center is a concentric pencil with the reflexion of the other limiting point as a center.*[1]

§ 3. **Circular Transformations.**[2] **Groups of Substitutions.** A linear transformation of a complex variable

[1] A great number of other theorems on reflexion may be found in J. CASEY: *A treatise on Analytic Geometry*, chapt. III (1893); R. STURM: *Die Lehre von den geometrischen Verwandtschaften*, Vol. IV, pp. 72–95 (1909); W. FIEDLER: *Cyklographie* (1882).

[2] A comprehensive study of such transformations may be found in an article by F. N. Cole on "The linear functions of a complex variable," *Annals of Mathematics*, Vol. V, pp. 121–176 (1890).

$$(4) \qquad z' = \frac{az + b}{cz + d},$$

has the well-known property that it transforms circles into circles (including straight lines) and is therefore frequently called a circular transformation. If we transform the point z' by another circular transformation

$$(5) \qquad z'' = \frac{a_1 z' + b_1}{c_1 z' + d_1},$$

then the relation between z'' and z is given by

$$(6) \qquad z'' = \frac{(aa_1 + cb_1)z + (ba_1 + db_1)}{(ac_1 + cd_1)z + (bc_1 + dd_1)},$$

that is, z'' results from z by another circular transformation. Denoting the first substitution by which z is replaced by z' symbolically by S, similarly the second, by which z' passes into z'', by T and the third by U, we may say that the product of the substitutions S and T is U, or that S followed by T gives U. This may be symbolically written:

$$ST = U.$$

If some other substitution R carries the point z'' into z''', then there is again a linear substitution (geometrically a circular transformation) which changes z directly into z'''. This substitution may be symbolically written in the form

$$STR = U.$$

In this combination of substitutions the associative law holds, that is,

$$S(TR) = (ST)R,$$

while, in general, the commutative law is not true; namely

$$ST \neq TS.$$

If S is given, then by the inverse S^{-1} of S we understand the substitution of

$$\frac{dz - b}{-cz + a}$$

for z, which is obtained by solving

$$z = \frac{az' + b}{cz' + d}$$

or z'. It is easily seen that SS^{-1} is a substitution which leaves z invariant and is therefore called the identical substitution and may be indicated by

$$SS^{-1} = 1.$$

The foregoing definitions are sufficient to convey the idea of the group. For the sake of brevity this discussion is confined to finite groups.

A finite group of substitutions consists of a finite number of substitutions such that the product of any two of these is again one of the substitutions. The inverse of any substitution of the group belongs to the group. The group contains the identical substitution.

Such a group is obtained, for example, if we take n substitutions each S with the property that

$$SSS \cdots S \equiv S^n = 1.$$

Now $1, S, S^2, S^3, \cdots, S^{n-1}$ form a finite group of order n, which is at the same time cyclic. The reason for the attribute "cyclic" is apparent from the arrangement of the substitutions of the group.

§ 4. **Involutory Transformations.** An involutory transformation, or simply an involution in a complex plane may be defined as

$$z' = \frac{az + b}{cz - a}. \tag{7}$$

From this we find

$$z = \frac{az' + b}{cz' - a}.$$

If we transform by (7) z into z' and by the same transformation

$$z'' = \frac{az' + b}{cz' - a},$$

z' into z'', we find $z'' = z$, that is, by repeating an involution twice in succession, the original point z is transformed back into itself. Denoting the involutory substitution (7) by T, we have therefore, according to § 3,

$$T^2 = 1, \tag{8}$$

so that $1, T$ are the substitutions of a finite group of order 2, which is of course cyclic.

We shall now prove

THEOREM 4. *An involution in a complex plane is an inversion with respect to a fixed circle and a fixed axis through its center.*[1]

For this purpose turn the point z and its involutoric $(az + b)/(cz - a)$ through an angle ϕ about the origin. To their new positions apply a translation α, so that after the combined motion the points will be in the positions

$$e^{i\phi}z + \alpha \quad \text{and} \quad e^{i\phi}\frac{az + b}{cz - a} + \alpha.$$

[1] In a different form this proposition is also stated and proved in HOLZMÜLLER: *Einführung in die Theorie der isogonalen Verwandtschaften*, pp. 43–46 (1882).

If it is possible to determine ϕ, α and a real positive quantity ρ such that

$$(9) \qquad (e^{i\phi}z + \alpha)\left(e^{i\phi}\frac{az+b}{cz-a} + \alpha\right) = \rho^2,$$

for all values of z, the proposition will be proved.

Equation (9) may be written in the form

$$e^{i\phi}(e^{i\phi}a + c\alpha) + \{e^{i\phi}(e^{i\phi}b - a\alpha) + \alpha(e^{i\phi} + c\alpha) - c\rho^2\}z + \alpha(e^{i\phi} - a\alpha) + a\rho^2 = 0.$$

We have therefore the conditions

$$e^{i\phi}a + c\alpha = 0,$$
$$e^{i\phi}(e^{i\phi}b - a\alpha) = c\rho^2,$$
$$\alpha(e^{i\phi}b - a\alpha) = - a\rho^2.$$

But the first of these is a consequence of the last two. From these is found

$$(10) \qquad \alpha = \pm\frac{a\rho}{\sqrt{a^2+bc}}, \qquad e^{i\phi} = \mp\frac{c\rho}{\sqrt{a^2+bc}}.$$

As the absolute value of $e^{i\phi}$ is 1, we find for ρ

$$(11) \qquad \rho = \left|\frac{\sqrt{a^2+bc}}{c}\right|,$$

which is easily recognized as half the distance between the double-points of the involution. From (9) and (10) it would appear that the problem has two solutions. To show that the inversion is unique take first in (9) and (10) for α and $e^{i\phi}$ the positive and negative sign and designate the corresponding values by α_1 and ϕ_1. Taking $\alpha = \alpha_1$ and $\phi = \phi_1$, then

$$e^{i\phi_1}\cdot z \quad \text{and} \quad e^{i\phi_1}\cdot\frac{az+b}{cz-a}$$

are inverse with respect to a circle having $-\alpha_1$ as a center, ρ as given by (11) as a radius, and the line l through $-\alpha_1$ parallel to the real axis as an axis. Rotating the figure of this inversion back through an angle $-\phi_1$ we obtain the original points z and $(az+b)/(cz-a)$ and $-\alpha_1$ is moved to a position $\beta = -\alpha_1 e^{i\phi}$. The axis l is moved to a position g through β. But $\beta = +(a/c)$ is the midpoint between the double-points. *Consequently* z *and* $(az+b)/(cz-a)$ *are inverse with respect to a circle through the double-points of the transformation as extremities of a diameter and with the line joining them as an axis.*

Taking for α the other sign $\alpha_2 = -\alpha_1$, and $\phi_2 = \phi_1 + \pi$, we get the same result.

§ 5. **General Circular Transformation of an Inversion and an Ordinary Reflexion.** It is well-known that the transformed of a group is an isomorphic

group, that is, a group which obeys the same formal laws as the original group. Hence if we transform the cyclic group of order 2

$$\left(z, \quad \frac{az+b}{cz-a}\right)$$

by any circular transformation

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta},$$

and designate the substitutions $(az + b)/(cz - a)$ and $(\alpha z + \beta)/\gamma z + \delta)$ by the symbols S and T respectively, the transformed group

$$(1, \quad T^{-1}ST)$$

is also of order 2, and is consequently geometrically represented by an involution, or an inversion on a circle.

This result may be stated as

THEOREM 5. *A circular transformation of an inversion is again an inversion.* In connection with theorem (1) we deduce at once

THEOREM 6. *A circular transformation of a reflexion on a circle is again a reflexion on a circle, or on a straight line.*[1]

Consider now the group of order 2,

$$(z, \quad 2\lambda - z),$$

where λ is a constant, and which geometrically is represented by an ordinary reflexion on a straight line parallel to the axis of imaginaries at a distance λ from the origin. The transformed of this group by any circular transformation is again of order 2 and is geometrically represented by a reflexion on the circle which is the transformed of the line $x = \lambda$ in the z-plane.

Thus we can state

THEOREM 7. *A circular transformation transforms a system of parallel equidistant straight lines into a parabolic pencil of circles, so that of any three consecutive circles C_1, C_2, C_3, we may consider C_1 and C_3 as inversions or reflexions on C_2.*

§ 6. **Geometric Properties of Ring-Shaped Domains Connected with Infinite Cyclic Groups of Linear Substitutions.** As is well known every linear substitution between two complex variables may be written in one of the two forms,

$$(12) \qquad \frac{x_1 - a}{x_1 - b} = q\frac{x - a}{x - b}, \qquad (13) \quad \frac{A}{x_1 - a} = \frac{A}{x - a} + 1,$$

where x_1 and x are the variables and a and b are the double-points. In (12) we have the loxodromic case when $|q| \lesseqgtr 1$, elliptic when $|q| = 1$. In (13) we have the parabolic substitution. I shall confine myself to the loxodromic and parabolic

[1] Corresponds to theorem 4, p. 244, Osgood, *loc. cit.*

cases. Repeating the substitutions (12) and (13) each λ times in succession we get

$$\frac{x_\lambda - a}{x_\lambda - b} = q^\lambda \frac{x - a}{x - b}, \tag{14}$$

$$\frac{A}{x_\lambda - a} = \frac{A}{x - a} + \lambda, \tag{15}$$

representing, when λ assumes all integral values between $-\infty$ and $+\infty$, cyclic groups. Consider first the loxodromic group, where $|q| \neq 1$. By the linear substitution

$$c \cdot \frac{x - a}{x - b} = z, \tag{16}$$

the x-plane is transformed into the z-plane, and the substitutions of the transformed group assume the simple form

$$z_\lambda = q^\lambda z. \tag{17}$$

As the fundamental domain of this group we choose the ring-shaped surface between the unit-circle and the concentric circle of radius $|q|$. The remaining domains are bounded by the pencil of concentric circles with radii $|q^\lambda|$, where $-\infty < \lambda < +\infty$. Since for any three consecutive circles C_λ, $C_{\lambda+1}$, $C_{\lambda+2}$ there is

$$|q^\lambda| \cdot |q^{\lambda+2}| = |q^{\lambda+1}|^2,$$

we conclude that C_λ and $C_{\lambda+2}$ are reflexions on $C_{\lambda+1}$. Transforming back into the x-plane the pencil (C_λ) is transformed into the elliptic pencil with a and b as limiting points. The transformed circles K_λ corresponding to those of C_λ form the boundaries of the domains belonging to the group (14). From theorem (6) we conclude immediately:

THEOREM 8. *Of any three consecutive circles K_λ, $K_{\lambda+1}$, $K_{\lambda+2}$ of the domains associated with the cyclic loxodromic group, K_λ and $K_{\lambda+2}$ are reflexions on $K_{\lambda+1}$.*

In case of the parabolic group we may transform it by the substitution

$$\frac{A}{x - a} + c = z \tag{18}$$

into the z-plane, so that we get

$$z_\lambda = z + \lambda. \tag{19}$$

As the fundamental region we choose the strip between the axis of imaginaries and a line parallel to it at a distance equal to unity. The domains are now bounded by a system of parallel equidistant lines. Hence according to theorem (7) in the corresponding pencil of the x-plane of any three consecutive circles K_λ, $K_{\lambda+1}$, $K_{\lambda+2}$ we find as in the foregoing theorem that K_λ and $K_{\lambda+2}$ are reflexions on $K_{\lambda+1}$. These properties which are well-known results[1] also follow from the theory of conformal transformations.

[1] See § 16, p. 200, Vol. 1, *Modulfunctionen*, by Klein-Fricke.